Equilibrium and Coordination

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- Many examples where current prices (and other state variables) are determined by individual expectations
- Question: expectations formation?
- Common assumption: rational expectations
  - = agents know the distribution of current and future prices (and other state variables)
  - Without uncertainty = perfect foresight

- A survey of some ideas discussing the assumption of RE
- Method: write the model as a "temporary equilibrium map"

$$p_t = f\left(\bar{p}^e_{t+1}\right)$$

where $\bar{p}^e_{t+1}$ is the average expectation of prices at $t+1$ (may include lagged variables p_{t-1}, expectations p^e_t or p^e_{t+2}, heterogenous expectations, multi-dimensional p_t, uncertainty on fundamentals,...)

Rational Expectations Equilibria

- Steady state: fixed point of f

$$p^* = f(p^*)$$

- RE do not exclude fluctuations:
- One example: 2-cycles (p_a, p_b)

$$p_a = f(p_b) \text{ and } p_b = f(p_a)$$

  - $=$ a fixed point of a map $(p_a^e, p_b^e) \mapsto (p_a, p_b)$:

$$(p_a, p_b) = (f(p_b^e), f(p_a^e))$$

Another example: stationary (markov) sunspot equ (=stochastic REE where the price process = Markov chain)

- 2-state ($s = a, b$) stationary Markov chain (=sunspot process)
- Sunspot equ = price perfectly correlated with sunspot (p_a, p_b)
- At t, agents observe the sunspot s and expect that prices at $t+1$ are p_a^e with proba π_{sa} and p_b^e with proba π_{sb}
- In the linear case $(f(\bar{p}^e) = \beta \bar{p}^e)$:

$$
\begin{aligned}
p_a &= \beta \left(\pi_{aa} p_a^e + \pi_{ab} p_b^e \right) \\
p_b &= \beta \left(\pi_{ba} p_a^e + \pi_{bb} p_b^e \right)
\end{aligned}
$$

- Sunspot equilibrium = a fixed point: $(p_a^e, p_b^e) = (p_a, p_b)$
- Existence (for some Markov chains) when $|\beta| > 1$
- Multiplicity = a coordination pb (expecting a REE is "rational" only if others expect the same REE (=Nash): not always rational to have rational expectations)

Coordination on Rational Expectations

- Rational expectations equilibrium price p^*: everyone expects $p^* \Rightarrow p^*$ occurs
- Robustness criterion: everyone expects the price p in a neighborhood of p^* (no additional info on p) $\Rightarrow$ the actual price is in this neighborhood of p^*
- Formally: the temporary equilibrium map $p = f(p^e)$ is contracting ($|f'| < 1$)
- Economic intuition: the price is not very sensitive to expectations

CK of rationality and model

- Extending the above robustness criterion (+ game theoretic fundations)

Common Knowledge of sth: sth is true, everyone knows that sth is true, everyone knows that everyone knows,...
= sth is public

CK and equilibrium:

- An outcome consistent with CK of rationality, model (= the temporary equilibrium map) and the price process = an equilibrium
  - Agents know that the price at $t+1$ will be p_{t+1}, they are rational. Hence, the actual price is $f(p_{t+1}) = p_t$ (f assumes rationality)
- (common) knowledge of p is important: CK of rationality and model characterizes a broader set than the set of equilibria (= set of rationalizable outcomes)

- Example: $f\left(\bar{p}^{e}\right)=2\bar{p}^{e}$
- Any p is rationalizable because p occurs whenever the average expectation is $\frac{1}{2}p$, which in turns occurs whenever the average expectation is $\frac{1}{4}p$,... up to infinity
- This infinite sequence of expectations $\left(\frac{1}{2}p, \frac{1}{4}p, \ldots\right)$ corresponds to the CK assumptions (Higher Order Beliefs)
    - p occurs
    - $\Leftarrow$ everyone is rational and believes that $\frac{1}{2}p$ occurs
    - $\Leftarrow$ everyone believes that: f is the true model, everyone is rational, everyone believes that $\frac{1}{4}p$ occurs
    - $\Leftarrow$ everyone believes in the previous step
    - ...

- Some prior knowledge on p can restrict the set of possible prices
- Example: $f(\bar{p}^e) = \frac{1}{2}\bar{p}^e$
  - a price p is "justified" by a sequence of higher order beliefs $(2p, 4p, ...)$
  - p occurs $\Leftarrow$ everyone believes that $2p$ occurs
  - $\Leftarrow$ everyone believes that everyone believes that $4p$ occurs
  - ...
- Assume CK that p is bounded
  - for $p \neq 0$, sequence $(p, 2p, 4p, ...)$ not consistent with this assumption
  - only one admissible price: $p = 0$ (rational expectations equ)
- NB: A big difference between knowledge and CK: sequence $(p, 2p, 4p, ...)$ consistent with the assumption "everyone knows that p is bounded"

Stability criterion relying on CK
("eductive" stability)

Assume (common) knowledge of rationality and model, no prior knowledge of the price (and then no prior knowledge of others' expectations). Can the equilibrium be predicted? Yes, if the equilibrium is the unique rationalizable outcome.

Local criterion: CK that p is in a neighborhood N of the equilibrium

- everyone expects in N and is rational $\Rightarrow$ p is in $f(N)$
- everyone knows the previous step $\Rightarrow$ everyone expects in $f(N)$ $\Rightarrow$ p is in $f^2(N)$
- ...
- Local stability obtains when $\lim f^k(N) = \{p^*\}$

If f is a contraction, then p^* is locally stable

NB: a more rigourous treatment of heterogeneity of (stochastic) expectations is possible, the conclusion remains true.

The global stability criterion is also defined:

- the REE is stable whenever it is the unique rationalizable outcome
- same criterion as above where N is the whole set of prices

Remark on another kind of robustness criterion: adaptive learning (real time learning)

- The game is repeated, past prices are observed
- Assume a specific rule of expectation formation (based on past observations)
- An example:

$$p_t^e = p_{t-1}^e + \alpha \left(p_{t-1} - p_{t-1}^e \right)$$

- Study the long run behavior (and the speed of convergence) of the system

$$\begin{aligned} p_t &= f\left(p_t^e\right) \\ p_t^e &= p_{t-1}^e + \alpha \left(p_{t-1} - p_{t-1}^e \right) \end{aligned}$$

- convergence of p_t to the REE p^* (fixed point of f) linked to the value of f' (the same condition $|f'| < 1$ for convergence $\forall \alpha$ in the example)

Leading example (Guesnerie 1992)

A partial equilibrium model

- a continuum of producers $i \in [0, 1]$
- every producer i makes a production decision q_i before the price is known: q_i maximizes the expected profit $p_i^e q_i - \frac{q_i^2}{2\sigma}$

$$q_i = \sigma p_i^e$$

  where p_i^e is the price expectation of i
- aggregate demand $D(p) = \delta_0 - \delta p$
- Market clearing writes

$$D(p) = \int q_i di$$

- The temporary equilibrium map is

$$p = D^{-1}\left(\sigma \int p_i^e di\right)$$

- Rational Expectations Equilibrium

$$p^* = D^{-1}(\sigma p^*)$$

- Stability

$$-\frac{\sigma}{D'(p^*)} < 1$$

- Intuition: small effect of expectations on p
  - small σ (small supply elasticity): small effect of expectational mistakes on production
  - large D' (large demand elasticity): small effect of aggregate production on p

The common knowledge story

- Starting point: $p \leq \delta_0/\delta$ (maximum price for a non zero demand)
- Step 1 (everyone is rational): $q = \sigma p^e \leq \sigma\delta_0/\delta$, and the aggregate production is smaller than $\sigma\delta_0/\delta$

$$D(p) \leq \sigma\delta_0/\delta$$

  hence $p \geq p_1$ (with $D(p_1) = \sigma\delta_0/\delta$)
- Step 2 (everyone knows everyone is rational): everyone knows step 1, so that $q = \sigma p^e \geq \sigma p_1$ and the aggregate production is larger than σp_1

$$D(p) \geq \sigma p_1$$

  hence $p \leq p_2$ (with $D(p_2) = \sigma p_1$)
- ...
- Step $2n$: everyone knows step $2n-1$ (that is: $p \geq p_{2n-1}$)... Hence $p \leq p_{2n}$ (with $D(p_{2n}) = \sigma p_{2n-1}$)
- ...

- CK of rationality and model $\Rightarrow$ p is in the limit interval $[\lim p_{2n-1}, \lim p_{2n}]$ (set of rationalizable outcomes)
- The equilibrium p^* is always in this interval
- Stability $=$ $[\lim p_{2n-1}, \lim p_{2n}]$ reduces to p^*

Asymetric information on financial markets

A very common model for the market of a risky asset (Grossman 1976)

- risky asset: current price p (publicly observed), unknown future value θ
- agents $i \in [0, 1]$, identical but differentially informed about θ

What is this about?

- about information transmission by prices
- REE assumes that the information is correctly revealed
- conditions for REE stability?

The decision of i is a demand x_i for the risky asset

- conditional to the information set I_i
- max a mean/variance criterion (=CARA utility)

$$E(w) - \frac{a}{2} Var(w)$$

with w the future wealth (today's wealth is w_0 invested either in the risky asset or in a safe asset - money -)

- This is

$$x_i(I_i, p) = \frac{E(\theta|I_i) - p}{a Var(\theta|I_i)}$$

(E linear, Var constant, see below)

- Market clearing is

$$\int x_i (I_i, p)\, di = \varepsilon,$$

where the noisy supply ε is not observed (ε normally distributed)

- Hence p depends on ε and the information sets of agents:
  - p reveals some information privately known by agents (partial revelation only because of ε)
  - extracting information from p requires to have some belief about the correlation between p and agents' information

- Are these beliefs correct?
  - yes = REE
  - = problem of beliefs coordination (my beliefs about $(\theta, p) \Leftarrow$ my beliefs about others' demand $\Leftarrow$ my beliefs about others' beliefs about (θ, p))
  - Example: a high price p reveals that demand for the asset is high... But demand is high because (i) everyone knows that θ is high; or (ii) everyone disregards his own private information and believes that a high p reveals that others know that θ is high?
  - (ii) = coordination failure because agents believe that p reveals some private information, which is impossible since no one uses his private information (agents' belief are wrong)
  - Condition for avoiding such failures?

Information of agents

- Every agent i observes a private signal

$$s_i = \theta + \beta_i$$

  where β_i is normally distributed (with mean 0), i.i.d.
- Information set of $i = (s_i, p)$
- If all the signals s_i were public, the total information would be θ

Assume that (θ, ε, p) is normally distributed (this assumption is self-fulfilling)

- demand x_i is linear
- p is linear in (θ, ε)

Temporary equilibrium map

- If everyone expects a linear price function (characterized by 3 parameters $(\bar{p}, c_{\bar{s}}, c_{\varepsilon})$)

$$p(\bar{s}, \varepsilon) = \bar{p} + c_{\bar{s}}\theta + c_{\varepsilon}\varepsilon$$

  then

  - demand is linear in (s_i, p)
  - the actual price is linear (characterized by 3 parameters $(\mathcal{T}\bar{p}, \mathcal{T}c_{\bar{s}}, \mathcal{T}c_{\varepsilon})$)

$$p = \mathcal{T}\bar{p} + \mathcal{T}c_{\bar{s}}\bar{s} + \mathcal{T}c_{\varepsilon}\varepsilon$$

- Temporary equilibrium map

$$\mathcal{T} : \begin{cases} IR^3 \to IR^3 \\ (\bar{p}, c_{\bar{s}}, c_{\varepsilon}) \mapsto (\mathcal{T}\bar{p}, \mathcal{T}c_{\bar{s}}, \mathcal{T}c_{\varepsilon}) \end{cases}$$

Rational Expectations Equilibrium
(Nash equilibrium of the game)

- a fixed point of $\mathcal{T}$
- $=$ a self-fulfilling distribution (θ, ε, p)
- there is a unique equilibrium

Stability

- $\Leftrightarrow \mathcal{T}$ is a contraction
- Stability iff
$$Var\,(\theta|p) > Var\,(\theta|s_i)$$
(Var does not depend on s_i and p)
- Extracting info from the price requires to expect the joint distribution (p, θ)
  - if agents expect this info to be precise, then their decisions are very sensitive to their expected distribution (p, θ)
  - $\Rightarrow$ the actual distribution (p, θ) is very sensitive to the expected distribution (p, θ)
  - Recall the general intuition: high sensitivity of outcome to expectations is detrimental to stability
- The ability of prices to transmit information is limited by coordination difficulties