About mean field games

Séminaire interne du Labex MME-DII

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Sommaire

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- This theory developed by Jean-Michel Lasry and Pierre-Louis Lions uses in a game theory framework the concept of mean field which is taken from statistical physics.
- Similar ideas have been developed independently by Huang-Caines-Malhamé.

Mean Field Games Theory

The typical model at the heart of the mean field games (MFG) theory is the following system of PDEs :

$$\begin{aligned} (HJB) \qquad & -\partial_t u - \frac{\sigma^2}{2}\Delta u + H(x, m, Du) = 0 \\ & u(T, x) = G(x, m(T)) \\ (Kolmogorov) \quad & \partial_t m - \frac{\sigma^2}{2}\Delta m - div(mD_pH(x, m, Du)) = 0 \\ & m(0, x) = m_0(x) \end{aligned}$$

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$$

Each player chooses his optimal strategy in view of the global (or macroscopic) informations that are available to him and that result from the actions of all players.

Mean Field Games Theory

The **heuristic** interpretation :
An agent controls the SDE

$$dX_t = \alpha_t dt + \sigma dB_t$$

where (B_t) is a standard Brownian motion in order to minimize

$$\mathrm{E}\left(\int_0^T \frac{1}{2} L(X_s, m(s), \alpha_s) ds + G(X_T, m(T))\right)$$

The optimal control is a feefback control given by :
$\alpha^\star(t,x) = -D_p H(x, m, Du)$.

Sommaire

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The asset manager does not have the sole aim of satisfying his current customers,
he likes to have a good relative performance among the asset managers.

The model considers a continuum of asset managers who at time $t = 0$ have the same unitary amount to manage. These managers will invest in risk-free and risky assets in creating their portfolio :

- θ : proportion invested in risky assets
- $1 - \theta$: proportion invested in risky-free assets with return r.

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Let $r + \tilde{\varepsilon}$ be the return of the risky assets, where $\tilde{\varepsilon}$ is a random variable assumed to be distributed normally (to be specified later).

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$$\mathrm{E}(u(X) + \beta \tilde{C})$$

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Let M be the cumulative distribution function of the weights θ and m the correspending distribution function.
One can define $\tilde{C}$: $\tilde{C} = 1_{\tilde{\varepsilon}>0} M(\theta) + 1_{\tilde{\varepsilon}\leqslant 0}(1 - M(\theta))$.

Agent of type ε

Agent of type ε considers the problem

$$\max_{\theta} \mathrm{E}_{\varepsilon}(u(1 + r + \theta\tilde{\varepsilon}) + \beta\tilde{C})$$

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$$= -exp(-\lambda(1 + r + \theta\varepsilon) + 1/2\lambda^2\theta^2\sigma^2)$$
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\end{aligned}$$

Hint : a random variable X log-Normale with parameters μ and σ has k momentum

$$\mathrm{E}(X^k) = e^{k\mu + k^2\sigma^2/2}$$

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$$-\lambda^2\sigma^2(\theta-\frac{\varepsilon}{\lambda\sigma^2})exp(-\lambda(1+r)-\lambda\theta\varepsilon+1/2\lambda^2\theta^2\sigma^2)+\beta m(\theta)C(\varepsilon)=0$$

where $C(x)=2(N(\frac{x}{\sigma})-\frac{1}{2})$, N being the cumulative distribution function of a normal variable $\mathcal{N}(0,1)$.

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where $C(x)=2(N(\frac{x}{\sigma})-\frac{1}{2})$, N being the cumulative distribution function of a normal variable $\mathcal{N}(0,1)$. Here θ is an $\theta(\varepsilon)$,so $\varepsilon\mapsto\theta(\varepsilon)$ transports distribution f (ε distribution) toward distribution m of θ :

$$m(\theta)\theta^{'}(\varepsilon)=f(\varepsilon)$$

Differential equation for $\varepsilon \mapsto \theta(\varepsilon)$

Multiply the first order condition by $\theta'(\varepsilon)$ and using

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moreover θ must satisfy $\theta(0) = 0$. There exists a unique function θ that verifies (1) with the two additional constraints :

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- $\theta(\varepsilon) > \theta_0(\varepsilon) = \dfrac{\varepsilon}{\lambda\sigma^2}$
- $\lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$

Second order condition

Let

$$\Gamma(\varepsilon,\theta) = -\lambda^2\sigma^2(\theta - \frac{\varepsilon}{\lambda\sigma^2}) exp(-\lambda(1+r) - \lambda\theta\varepsilon + 1/2\lambda^2\theta^2\sigma^2) + \beta m(\theta) C(\varepsilon)$$

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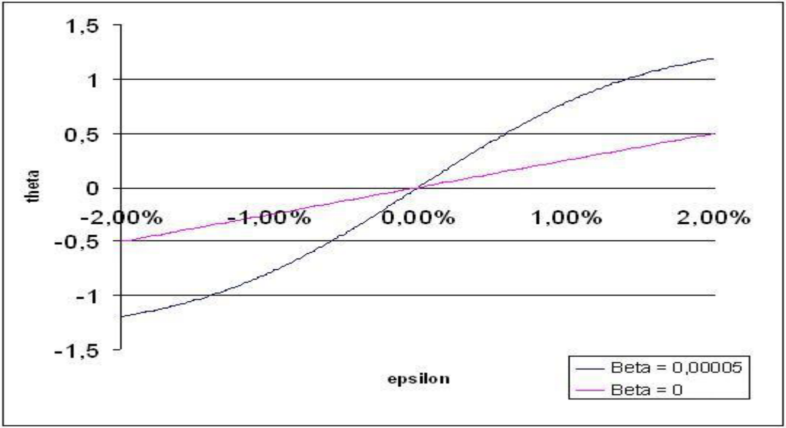
The second order condition is verified and $\theta(\varepsilon)$ characterize the maximum of the optimization problem.


1,5
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-1
-1,5
theta
-2,00%
-1,00%
0,00%
1,00%
2,00%
epsilon
Beta = 0,00005
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Sommaire

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Thus, for the agent i the problem is :

$$\sup_{(\alpha_s)_{s\geqslant 0}, X_0^i = x} \mathrm{E}\left[\int_0^\infty \left(g(s, X_s^i, m) - \frac{|\alpha(s, X_s^i)|^2}{2}\right) e^{-\rho s} ds\right]$$

$$dX_t^i = \alpha(t, X_t^i)dt + \sigma dW_t^i \text{ sur } [0, T]$$

where m is the distribution of agents in the social space and the function g will model the resemblance.

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- $(g(t, x, m) = \ln(m(t, x))$

Consider the case where $g(t, x, m) = \ln(m(t, x)$, the MFG problem leads to the PDEs system :

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\begin{aligned}
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The MFG "forward/backward" reasoning.

Stationary solutions

Proposition

Suppose that $\rho < \frac{2}{\sigma^2}$*. There exist three constants* $s^2 > 0$*,* $\eta > 0$ *and* w *such that* $\forall \mu \in \mathbb{R}^n$*, if* m *is the probability distribution function associated to a gaussian variable* $\mathcal{N}(\mu, s^2 I_n)$ *and* $u(x) = -\eta|x-\mu|^2 + w$*, then* (u, m) *is a stationary solution of the MFG problem.*

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These three constants are given by :

- $s^2 = \frac{\sigma^4}{4-2\rho\sigma^2}$
- $\eta = \frac{1}{\sigma^2} - \frac{\rho}{2} = \frac{\sigma^2}{4s^2}$
- $w = -\frac{1}{\rho}(\eta n \sigma^2 - \frac{n}{2}\ln(\frac{2\eta}{\pi\sigma^2}))$

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- The theorems on existence and uniqueness of MFG problem (JJM) do not apply to this case.
- Although the authors do not prove uniqueness they study what they called the eductive stability of solution.
- This "eductive stability" notion is used to design a numerical method for the MFG system of this system.

Sommaire

Existence theorems, ergodic theorems, the link between the asymptotic of "N-players" games and MFG

Existence theorems, ergodic theorems, the link between the asymptotic of "N-players" games and MFG represent a fast growing area.

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Consider the system of second order MFG

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\begin{aligned}
(HJB) \qquad & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m(t,\cdot)) \text{ in } \mathbb{R}^d \times (0,T) \\
(Kolmogorov) \quad & \partial_t m - \Delta m - div(mDu) = 0 \text{ in } \mathbb{R}^d \times (0,T) \\
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Let

$$P_1 = \{m \text{ Borel probability measure on } \mathbb{R}^d \text{ such that } \int_{\mathbb{R}^d} |x| dm(x) < +\infty\}$$

be endowed with the so called Kantorovitch-Rubinstein distance :

$$d(\mu,\nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \left(\int_{\mathbb{R}^{2d}} |x-y| d\gamma(x,y) \right)$$

Assumptions on F, G and m_0

There exists $C_0 > 0$ such that

- F and G are uniformly bounded by C_0 over $\mathbb{R}^d \times P_1$,

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- F and G are uniformly bounded by C_0 over $\mathbb{R}^d \times P_1$,
- $\forall (x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times P_1$

$$|F(x_1, m_1) - F(x_2, m_2)| \leqslant C_0(|x_1 - x_2| + d_1(m_1, m_2))$$

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- $\forall (x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times P_1$

$$|G(x_1, m_1) - G(x_2, m_2)| \leqslant C_0(|x_1 - x_2| + d_1(m_1, m_2))$$

Assumptions on F, G and m_0

There exists $C_0 > 0$ such that

- F and G are uniformly bounded by C_0 over $\mathbb{R}^d \times P_1$,
- $\forall (x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times P_1$

$$|F(x_1, m_1) - F(x_2, m_2)| \leqslant C_0(|x_1 - x_2| + d_1(m_1, m_2))$$

- $\forall (x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times P_1$

$$|G(x_1, m_1) - G(x_2, m_2)| \leqslant C_0(|x_1 - x_2| + d_1(m_1, m_2))$$

- The probability measure m_0 is absolutly continuous with respect to the Lebesgue measure, has a Hölder continuous density (m_0) s.t. $\int_{\mathbb{R}^d} |x|^2 m_0(x)dx < +\infty$

Theorem

Lasry-Lions 2006
Under the above assumptions, there exist a classical solution (u, m) *of (MFG).*

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Under the above assumptions, there exist a classical solution (u, m) *of (MFG).*$(u, m) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *are continuous, of class* C^2 *in space and* C^1 *in time*

Proof

Let $\mathcal{C}$ be the set of $\mu \in C_0([0,T],P_1)$ such that (for $C_1 > 0$)

$$\sup_{s \neq t} \frac{d_1(\mu(s),\mu(t))}{|t-s|^{1/2}} \leqslant C_1$$

Proof

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$\mathcal{C}$ is a convex, compact subset of $C_0([0, T], P_1)$.
The proof is done by constructing a continuous map $\Psi : \mathcal{C} \to \mathcal{C}$.

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$$-\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, \mu(t)) \text{ in } \mathbb{R}^d \times (0, T)$$
$$u(T, x) = G(x, \mu(T)) \text{ in } \mathbb{R}^d$$

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- Define $m = \Psi(\mu)$ as the solution of

$$\partial_t m - \Delta m - div(mDu) = 0 \text{ in } \mathbb{R}^d \times (0, T)$$
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- Ψ is well defined and continuous.

Proof

One conclude by Schauder fixed point theorem : the map $\Psi : \mu \mapsto m = \Psi(\mu)$ has a fixed point in $\mathcal{C}$, which is a solution of the MFG system.

Uniqueness

Assume further that

$$\int_{\mathbb{R}^d} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) > 0 \forall m_1, m_2 \in P_1, m_1 \neq m_2$$

and

$$\int_{\mathbb{R}^d} (G(x, m_1) - G(x, m_2)) d(m_1 - m_2)(x) \geqslant 0 \forall m_1, m_2 \in P_1.$$

Then there is a unique classical solution for the MFG system.

Applications to games with finitly many players

The classical solution (u, m) of the MFG system allows the construction of an ε-Nash equilibrium in the following game $\mathcal{J}_1^N, \cdots, \mathcal{J}_N^N$:
the player i is controlling :

$$dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i$$

with X_0^i is random and has for law m_0, and faces the minimization problem : : $\min \mathcal{J}_i^N(\alpha^1, \cdots, \alpha^N)$ where

$$\mathcal{J}_i^N(\alpha^1, \cdots, \alpha^N) =$$

$$\mathrm{E}\left(\int_0^T (\frac{1}{2}|\alpha_s^i|^2 + F\left(X_s^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{X_s^j} \right))ds + G(X_T^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{X_T^j} \right)$$

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All the X_0^i and the brownian (B_t^i) are independant.

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Applications to games with finitly many players

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Player i can choose his control adapted to the filtration $(\mathcal{F}_t = \sigma(X_0^j, B_s^j, s \leqslant t, j = 1, \cdots, N))$.
Let (u, m) be the classical solution of the MFG, set $\overline{\alpha}(t,x) = -D_x u(t,x)$ and consider the open-loop strategy $\tilde{\alpha}^i$ obtained by solving

$$d\overline{X}_t^i = \overline{\alpha}(t, \overline{X}_t^i)dt + \sqrt{2}dB_t^i$$

with random initial condition X_0^i and setting $\tilde{\alpha}_t^i = \overline{\alpha}(t, \overline{X}_t^i)$.

Applications to games with finitly many players

Theorem

Huang, Caines, Malhamé 2006
For any $\varepsilon > 0$, there is some N_0 such that if $N > N_0$, then the symmetric strategy $(\tilde{\alpha}^1, \cdots, \tilde{\alpha}^N)$ is an ε-Nash equilibrium in the game $\mathcal{J}_1^N, \cdots, \mathcal{J}_N^N$:

$$\mathcal{J}_i^N(\tilde{\alpha}^1, \cdots, \tilde{\alpha}^N) \leqslant \mathcal{J}_i^N((\tilde{\alpha}^j)_{j \neq i}, \alpha) + \varepsilon$$

for any control α adapted to the filtration $(\mathcal{F}_t)$ and any $i \in \{1, \cdots, N\}$.

Applications to games with finitly many players

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is largly open.

This is proved only for the stationnary case (Lasry-Lions 07) and conjectured in the general case.

Consider the system of local second order MFG

$$
\begin{array}{ll}
(HJB) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m(t,\cdot))\mathbb{R}^d \times (0,T) \\
(Kolmogorov) & \partial_t m - \Delta m - div(mDu) = 0\mathbb{R}^d \times (0,T) \\
 & m(0,x) = m_0(x), u(T,x) = u_f(x, m(T))
\end{array}
$$

where data are periodic in space and $F : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ is smooth.

Theorem

Cardaliaguet-Lasry-Lions-Porretta 2012
Assume that $F : \mathbb{R}^d \times [0, \infty[\to \mathbb{R}$ *is* C^1, $\mathbb{Z}^d$ *periodic in* x *and bounded below.*
Then there exist a classical solution (u, m) *of (MFG). It is unique if* F *is increasing.*

Sommaire

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MFG applications

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