Probabilistic Opinion Pooling

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drawing on work with Christian List (LSE) and work in progress with Marcus Pivato (Cergy-Pontoise)
The problem of group agency:

treating a group as an agent, with its own beliefs, desires, intentions, actions, plans, ...
Example: the House of Lords
The aggregative approach:

Group attitudes – beliefs, preferences, goals, … – are not disconnected from people’s attitudes, but a function (‘aggregation’) of them
My focus today: beliefs / opinions

Problem: how merge people’s conflicting opinions into group opinions?
Testing your intuition 1

- Ann and Bob wonder whether it will rain tomorrow (they will go hiking).
- Assumption:
  - Ann thinks it will rain with 90% probability;
  - Bob thinks it will rain with 80% probability.
- How probable is rain overall, i.e., from the group’s perspective?
Testing your intuition 2

● New start!
● Assumption:
  – Ann thinks rain is 9 times more likely than no rain;
  – Bob thinks rain is 4 times more likely than no rain;
● How much more likely is rain than no rain overall?
Oh!

• The two scenarios are equivalent: Rain is 9 (4) times more likely than no rain if and only if rain is 90% (80%) likely.
• So: If your first answer was ‘rain is 85% probable’, your second answer should be ‘rain is $\frac{85\%}{15\%} \approx 5.666...$ times more likely than no rain’.
Lesson:

if group beliefs are averages of individual beliefs, then it’s unclear what exactly to average
Next test question

- After Ann and Bob tell each other their subjective probabilities of rain (90% and 80%), Ann says:

"Look, Bob. We both came up independently with high probabilities of rain. This is double confirmation for rain! The overall probability of rain must be at least 95%.

- Who agrees with Ann?
N.B.: exaggerating rather than averaging

- If Ann is right, then the group’s overall opinion (probability) should be **more extreme** than each group member’s opinion – rather than some **average** or **compromise** between the group members’ opinions.
Is Ann right?

This depends, among other things, on Ann’s and Bob’s sources of information:

- If both drew on similar information (e.g., both heard the same weather forecast), there is no real ‘double-confirmation’, and Ann is wrong.
- If both drew on independent information, Ann might be right.
Bottom line:

Opinion pooling is non-trivial !!
Three rival approaches

- What should a group’s overall probability of a scenario be?
- Three potential answers:
  - an **arithmetic average** of people’s probabilities (‘linear pooling’)
  - a **geometric average** of people’s probabilities (‘geometric pooling’)
  - a **product** of people’s probabilities (‘multiplicative pooling’)\(^1\)

\(^1\)The exact definitions of geometric and multiplicative pooling involve slightly more than taking a geometric average or product. Details soon!
Opinion pooling formalized
The individuals

A group of $n \geq 2$ individuals, labelled $i = 1, \ldots, n$, who have to assign collective probabilities to some relevant events.
The scenarios (‘possible worlds’)  

- $\Omega$: set of possible worlds/states/scenarios/... ($\Omega$ is non-empty and finite)  
- Examples:  
  - $\Omega = \{\text{rainy, not-rainy}\}$  
  - $\Omega = \{\text{rainy, cloudy, bright}\}$  
  - $\Omega = \{\text{Hollande wins next presidential election, Sarcozy wins, someone else wins}\}$  
  - $\Omega = \{0, 1, ..., m\}$, where a $\omega$ in $\Omega$ represents the number of students coming to my office hours next week  
  - ...
Probability functions

- An individual’s beliefs/opinions are captured by his probability function.
- A probability function is a function $P$ which maps each scenario $\omega$ in $\Omega$ to a probability $P(\omega) \geq 0$ such that the total probability is $\sum_{\omega \in \Omega} P(\omega) = 1$. 
**Aggregation (pooling)**

- A combination of opinion functions across the $n$ individuals, $(P_1, ..., P_n)$, is called a profile.
- A **pooling function** or **aggregation function** is a function $F$ which transforms any profile $(P_1, ..., P_n)$ of individual probability functions into a single collective probability function $P = F(P_1, ..., P_n)$, often denoted $P_{P_1, ..., P_n}$.
Linear pooling ("arithmetic averaging")

- The pooling function is **linear** if, for every profile \((P_1, \ldots, P_n)\), the collective probability of each scenario \(\omega\) is a weighted arithmetic average

\[
P_{P_1, \ldots, P_n}(\omega) = w_1P_1(\omega) + \cdots + w_mP_n(\omega)
\]

of people’s probabilities, for some fixed weights \(w_1, \ldots, w_n \geq 0\) of sum 1.

- **Extreme case:** If \(w_i = 1\) for some ‘expert’ \(i\) and \(w_j = 0\) for all other individuals \(j\), then we obtain an ‘expert rule’ given by \(P_{P_1, \ldots, P_n} = P_i\).
Geometric pooling ("geometric averaging")

- The pooling function is **geometric** if, for every profile \((P_1, ..., P_n)\), the collective probability of each scenario \(\omega\) takes the form

\[
P_{P_1, ..., P_n}(\omega) = c[P_1(\omega)]^{w_1} \cdots [P_n(\omega)]^{w_n}
\]

where \(w_1, ..., w_n\) are fixed non-negative weights with sum 1 and \(c\) is a scaling factor, given by

\[
c = \frac{1}{\sum_{\omega' \in \Omega}[P_1(\omega')]^{w_1} \cdots [P_n(\omega')]^{w_n}}.
\]

- **Extreme case:** If \(w_i = 1\) for some 'expert' \(i\) and \(w_j = 0\) for all other individuals \(j\), then we again obtain an 'expert rule' given by \(P_{P_1, ..., P_n} = P_i\).
- Geometric pooling assumes that \(\bigcap_{i} \text{supp}(P_i) \neq \emptyset\) (to ensure \(c\) is well-defined).
Multiplicative pooling

- The pooling function is **multiplicative** if, for every profile \((P_1, \ldots, P_n)\), the collective probability of each scenario \(\omega\) takes the form

\[
P_{P_1,\ldots,P_n}(\omega) = cP_1(\omega) \cdots P_n(\omega)
\]

where \(c\) is a scaling factor, given by

\[
c = \frac{1}{\sum_{\omega' \in \Omega} P_1(\omega') \cdots P_n(\omega')}.
\]

- Multiplicative pooling assumes that \(\cap_i \text{supp}(P_i) \neq \emptyset\) (to ensure \(c\) is well-defined).
Which of the three pooling methods — linear, geometric, multiplicative — is best?

The *axiomatic method* can help us give answers!
The "indifference preservation" axiom

- The axiom informally: If each individual finds all scenarios equally likely, then so does the group.
- The axiom formally: If each of $P_1, ..., P_n$ is the uniform probability distribution, so is $P_{P_1, ..., P_n}$.

$\Rightarrow$ Satisfied by linear, geometric and multiplicative pooling
The "consensus preservation" axiom

- The axiom informally: If all individuals agree, i.e., have the same beliefs, then these shared beliefs become the collective beliefs.
- The axiom formally: If $P_1 = \cdots = P_n = P$, then $P_{P_1, \ldots, P_n} = P$.

=> Plausible?
=> Satisfied by linear and geometric pooling, not multiplicative pooling
The axiom of "scenarios-wise pooling"

- The axiom informally: The group’s probability of a scenario is determined by people’s probabilities of this scenario (irrespective of people’s probabilities of other scenarios).
  - So the group’s probability of "rain" depends only on people’s probabilities of rain, not on people’s probabilities of "clouds", "sunshine", "hail", "snow", ...
- The axiom formally: The collective probability of a world \( \omega \) is expressible as a function of people’s probabilities of this world, \( P_1(\omega), ..., P_n(\omega) \).

=> Plausible?
=> Satisfied by linear pooling only
The axiom of "Bayesianity"

- The axiom informally: If all individuals learn the same event, then group beliefs change by conditionalisation on this event.
- The axiom formally: for any event $E \subseteq \Omega$ (consistent with profile), $P_{P_1^E, \ldots, P_n^E} = P_{P_1^E, \ldots, P_n}$. 

$\Rightarrow$ Satisfied by geometric and multiplicative pooling
The axiom of "external Bayesianity"

- This axiom concerns learning a likelihood function \( L : \Omega \rightarrow (0, \infty) \), not an event.
- Example: Learning statistical data in Bayesian statistics with multiple statisticians, where \( \Omega \) is the parameter space.
- For a probability function \( P \) and a likelihood function \( L : \Omega \rightarrow (0, \infty) \), we write \( P^L \) for the posterior probability function conditional on \( L \), defined as the unique probability function which, as a function of worlds, is proportional to \( P \cdot L \).

- The axiom informally: If all individuals learn a likelihood function, then group beliefs change by conditionalisation on it.
- The axiom formally: for any likelihood fn. \( L \), \( P_{P_1^L, \ldots, P_n^L} = (P_{P_1, \ldots, P_n})^L \).

\( \Rightarrow \) Satisfied by geometric pooling only
The axiom of "individual-wise Bayesianity"

- Assume only one individual learns the information.
  \[ \Rightarrow \text{idea: people have access to different sources of information ('informational asymmetry')} \]

- The axiom informally: It someone learns a likelihood function, group beliefs change be conditionalisation on it.

- The axiom formally: For any information \( L \) and individual \( i \), \( P_{P_1,\ldots,P_i^L,\ldots,P_n} = (P_{P_1,\ldots,P_n})^L \).

\[ \Rightarrow \text{Plausible?} \]
\[ \Rightarrow \text{Satisfied by multiplicative pooling only} \]
## Summary

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<th>Linear</th>
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<th>Multiplicative</th>
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Three theorems

**Theorem 1.** (Aczél-Wagner 1980, McConway 1981) The linear pooling functions are the only consensus-preserving and scenario-wise pooling functions (assuming \( \Omega \) contains more than two scenarios).

**Theorem 2.** (Genest 1984) The geometric pooling functions are unanimity-preserving and externally Bayesian (but there exist other pooling functions with these two properties).

**Theorem 3.** (Dietrich-List 2014) The multiplicative pooling function is the only indifference-preserving individual-wise Bayesian pooling function.
Our question "which of the three pooling functions is best?" has been reduced to another question: "which axioms are appropriate?"
Which pooling functions and axioms are right?

=> Goal-dependence

- Do we pursue an **epistemic or procedural** goal?
- I.e., should group opinions ‘track the truth’ or ‘track people’s opinions’?

Under the epistemic goal, linear pooling looks bad:
- It can’t handle information-learning! Neither externally nor individual-wise Bayesian.
Which pooling function and axioms are right?

$\Rightarrow$ Context-dependence

Suppose the goal is epistemic: we aim for *true* group beliefs!

How to aggregate depends crucially on the informational setting:

Case 1: **symmetric information:** all individuals base their beliefs on the same information

$\Rightarrow$ Here, geometric pooling and external Bayesianity are plausible.

Case 2: **asymmetric information:** each individual holds private information

$\Rightarrow$ Here, multiplicative pooling and individual-wise Bayesianity are plausible.
Cases between public and private information
Informational axioms

• "If information is learnt, group beliefs change by conditionalisation on it"
• There are $2 \cdot 3 = 6$ variants of this axiom, depending on
  – what is being learnt, i.e., an event or a likelihood function,
  – who learns the information, i.e., everyone (public info), a single individual (private info) or an arbitrary subgroup.
Conjectures
Bottom line: It depends!

Before pooling opinions, one must know

(i) the goal,

(ii) the informational context.
Thanks!